NAVORD REPORT 2772

AN ANALYSIS OF EDDY CURRENT LOSS IN LAMINATED CORE MATERIAL

2 JUNE 1953

U. S. NAVAL ORDNANCE LABORATORY
WHITE OAK, MARYLAND

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ABSTRACT: The eddy current loss in a laminated, constant permeability (i.e. non-ferromagnetic) core material is derived rigorously starting with Maxwell's equations. By comparing the results of this rigorous derivation with the expression commonly given in the literature, the limitations of the popular expression become immediately apparent. It is also emphasized that neither the popular nor the rigorous results can be applied properly to the ferromagnetic case, although such an application is commonly referred to in the literature.

U. S. NAVAL ORDNANCE LABORATORY
White Oak, Silver Spring, Maryland

NAVORD Report 2772

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This report has been prepared under task NOL-Re4a-56-1-53 in connection with the analysis of power losses in magnetic materials. It is for information only and not intended as a basis for action.

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INTRODUCTION

1. The popular expression given in the literature (Refs. 1, 2, 3) for the eddy current power loss per unit volume at low frequencies in ferromagnetic laminations which form the core of a long solenoid or a toroid is

$$\frac{P}{V} = \frac{1}{6} K \pi^2 B_m^2 f^2 d^2 \sigma ; \qquad \text{(I)}$$

the definition of the symbols will be given in the text of this report. This expression is derived essentially by finding the average effective current density from the assumption of an average effective magnetic field. The line integral of the electric intensity vector about a chosen area set equal to the time rate of change of the flux passing through the area, plus the use of Ohm's law, enables one to find the average effective current density. Squaring this current density and dividing by the conductivity yields the power loss expression given above.

2. A more rigorous solution to the eddy current power loss problem for a non-ferromagnetic [illegible] constant permeability can be obtained by solving Maxwell's equations. This solution for the power loss per unit volume obtained by this method, and derived in detail below, is

$$\frac{P}{V} = \frac{1}{6} K \pi^2 \overline{B}_m^2 f^2 d^2 \sigma \quad \text{for } 2\pi d \sqrt{\sigma \mu f} < 3 \qquad \text{(II)}$$

$$\frac{P}{V} = \pi K \overline{B}_m^2 f^{\frac{3}{2}} \sigma^{\frac{1}{2}} d^{-1} \quad \text{for } 2\pi d \sqrt{\sigma \mu f} > 3 \qquad \text{(III)}$$

A comparison of the popular expression (I) with the rigorous expressions (II) and (III), derived for the non-ferromagnetic case, shows, among other things, that (I) also cannot be properly applied to the ferromagnetic case. Other results of the comparison are given in detail below.

3. The following material in this report contains a derivation of the eddy current power loss per unit volume, as given by the above expressions for a laminated core having a constant permeability and negligible hysteresis effects.

FORMULATION OF THE PARTIAL DIFFERENTIAL EQUATION FROM MAXWELL'S EQUATIONS

4. The variables associated with an electro magnetic field in any lamination of dimensions large compared to atomic dimensions must satisfy the following two Maxwell's equations

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \qquad \text{(Rationalized M.K.S. Units)} \quad (1)$$

$$\nabla \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t} \qquad \text{(Rationalized M.K.S. Units)} \quad (2)$$

here $\overline{E}$ = electric intensity vector
$\overline{D}$ = electric displacement vector
$\overline{B}$ = magnetic induction vector
$\overline{H}$ = magnetic intensity vector
$\overline{J}$ = current density vector

If we assume that

$$\overline{B} = \mu \overline{H} = k_m \mu_0 \overline{H} \quad (3)$$

$$\mu = k_m \mu_0 = \text{constant} \quad (4)$$

$$\overline{J} = \sigma \overline{E} \quad (5)$$

where

K_m = the permeability
μ_0 = $4\pi \times 10^{-7}$ henry/meter
σ = specific conductivity
ϵ = $D/E = K_e \epsilon_0$
K_e = dielectric constant
ϵ_0 = 8.85×10^{-12} farad/meter
ω = Angular frequency = $2\pi f$

equations (1) and (2) can be combined by use of (3) and (5) to give the partial differential equation.

$$\nabla^2 \overline{H} - \sigma\mu \frac{\partial \overline{H}}{\partial t} - \epsilon\mu \frac{\partial^2 \overline{H}}{\partial t^2} = 0 \quad (6)$$

For frequencies at which $\sigma/\omega \gg \epsilon$ equation (5) becomes

$$\nabla^2 \overline{H} = \sigma\mu \frac{\partial \overline{H}}{\partial t} \quad (7)$$

This equation in e.m.u. is

$$\nabla^2 \bar{H} = 4\pi\sigma\mu \frac{d\bar{H}}{dt} \tag{8}$$

To obtain the boundary conditions let us first inspect figure 1 which is a sketch of part of an infinite solenoid having a square cross section and containing thin rectangular laminations which have their longitudinal axes parallel to the axis of the solenoid. If a line integral of the magnetic intensity is taken around the path a b c d a, which is in a plane that lies between the laminations, we see that just the line integral from a to b is of importance since the magnetic intensity is perpendicular to paths ad and bc and the magnitude of the field outside of the solenoid is zero. Because of symmetry, the field at any point along ab is constant and its magnitude is determined by the total current passing through the area abcd. For low frequencies at which displacement currents between the laminations are negligible, the current passing through the area is Ni where N is the number of turns enclosed by area abcd and i is the instantaneous value of current passing through the wire. Hence from Ampere's circuital law in rationalized M.K.S. units we get for the magnetic field intensity H_B between the laminations

$$H_B = n i \tag{9}$$

where n is the number of turns per unit length of the solenoid. Because the field at any point is determined solely by ni, the boundary condition for (7) is that the instantaneous value of the magnetic intensity at any point on the boundary of any lamination is the same as that at any other boundary point and is given by (9).

5. Figure 2 is a sketch of a section of one lamination. The origin of the coordinates is chosen so that the plane y = o lies midway between the two largest parallel surfaces of the lamination or, in other words, these surfaces are at y = + T where T is the half thickness of the lamination. If the Z axis of the lamination is parallel to the solenoid axis, the magnetic intensity will have only a Z component. The penetration of magnetic intensity into the lamination to points not too close to x = + W will come mostly from the y = + T surfaces because of the shorter penetrating distances involved. From (2),

$$\bar{J} = \nabla \times \bar{H} = \bar{i}\frac{dH_z}{dy} - \bar{j}\frac{dH_z}{dx} \tag{10}$$

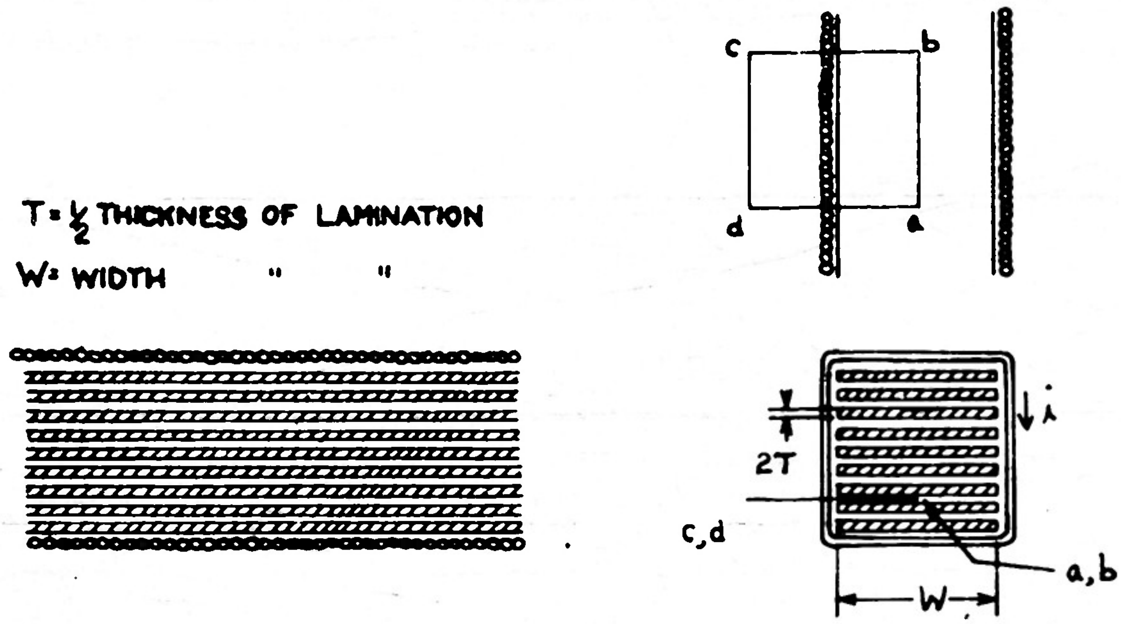


Fig I - SECTION OF A LONG SOLENOID CONTAINING A LAMINATED CORE

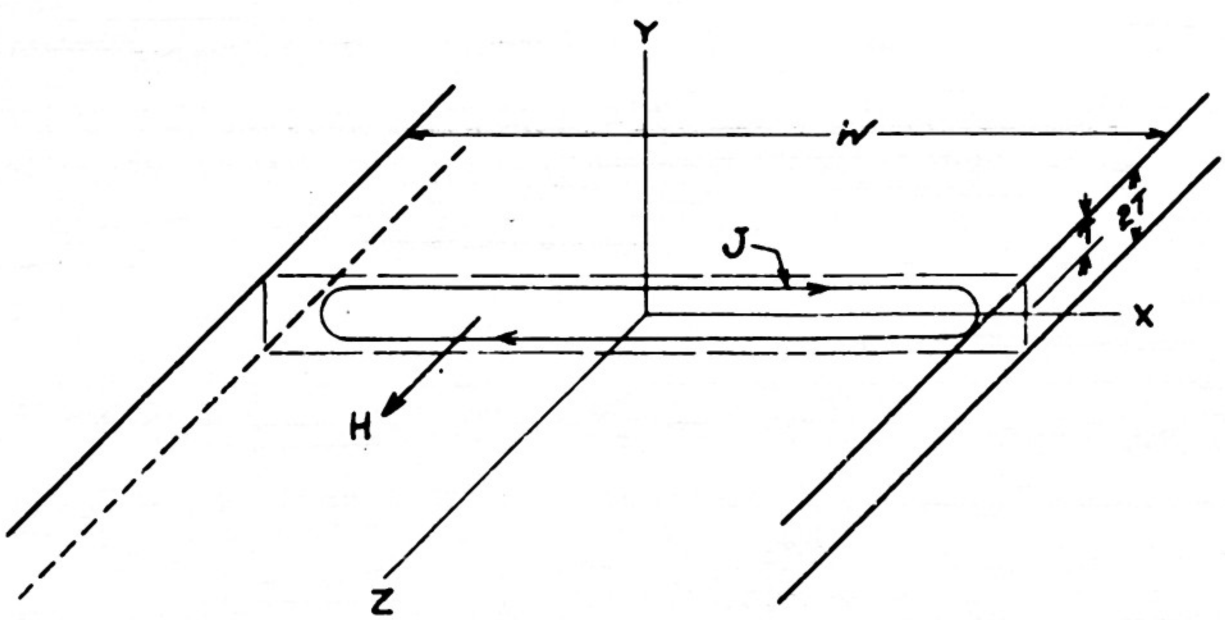


Fig. 2 - SECTION OF A LAMINATION

Consequently, if the lamination thickness is small compared to the width, the second term of the right side of (10) becomes relatively small compared to the first term for most points in the lamination. Hence, the current density essentially has just an "X" component except near the edges, $X = \pm W/2$, and the current path for constant density amplitude will look something like that shown in figure 2. Another consequence of the intensity being principally a function of y is that the Laplacian of the intensity of (7) becomes approximately

$$\frac{\partial^2 H_z}{\partial y^2} = \sigma \mu \frac{\partial H_z}{\partial t} \qquad (11)$$

Since the magnetic intensity at points inside of a lamination is a complex function of y and since the magnetic intensity penetrates the lamination from both sides ($y = \pm T$) in a similar manner, $\partial H_z/\partial y$ at $y = o$ must either be zero, infinity or discontinuous. However, $\partial H/\partial y$ also gives the magnitude of the current density so that the condition that agrees with the physical case is that

$$\frac{\partial H_z}{\partial y} = 0$$

at $y = o$. This is the second condition that the solution of (7) must satisfy.

SOLUTION OF THE PARTIAL DIFFERENTIAL EQUATION

6. The solution of (11) which satisfies the required conditions can be found by the usual separation-of-variables method if μ is constant and a sinusoidal current is passed through the windings. Such a solution is of the form

$$H_z = A\left(e^{\alpha y} + e^{-\alpha y}\right)e^{j\omega t} = 2A\cosh \alpha y \, e^{j\omega t} \qquad (12)$$

where $$\alpha = \sqrt{\frac{\sigma\mu\omega}{2}}\,(1+j)$$

If we take the derivative of H_z with respect to y we see that $\partial H/\partial y = 0$ AT $y = 0$. AT $y = \pm T$, $H_z = H_o$ OR, FROM (9)

$$H_z = ni = nIe^{j\omega t} = 2Ae^{j\omega t}\cosh \alpha T \qquad (13)$$

and $$A = \frac{nI}{2\cosh \alpha T} \tag{14}$$

where I is the amplitude of the sinusoidal current passing through the winding. The complete approximate solution of (7) for the case of laminations of thickness small compared to the width is then

$$H_z = nI \frac{\cosh \alpha y}{\cosh \alpha T} e^{j\omega t} \tag{15}$$

The real test of this solution is that we should be able to find the current density by two different ways from Maxwell's equations. The first method uses (1), (3) and (5) as follows:

$$\nabla \times \bar{E} = -\bar{K} \frac{dE_x}{dy} = -\frac{\partial \bar{B}}{\partial t} = -\bar{K} j\omega\mu n I \frac{\cosh \alpha y}{\cosh \alpha T} e^{j\omega t}$$

$$= -\frac{\bar{K}}{\sigma} \frac{dJ_x}{dy} \tag{16}$$

$$\frac{dJ_x}{dy} = j\omega\sigma\mu n I \frac{\cosh \alpha y}{\cosh \alpha T} e^{j\omega t}$$

$$J_x = \frac{j\omega\sigma\mu n I}{\alpha} \frac{\sinh \alpha y}{\cosh \alpha T} e^{j\omega t} = \alpha n I \frac{\sinh \alpha y}{\cosh \alpha T} e^{j\omega t}$$

We can also find J by using just (2), ie,

$$\bar{J} = \nabla \times \bar{H} = \bar{T} \frac{dH_z}{dy} = \bar{T} \alpha n I \frac{\sinh \alpha y}{\cosh \alpha T} e^{j\omega t} \tag{17}$$

$$J_x = |\bar{J}| = \alpha n I \frac{\sinh \alpha T}{\cosh \alpha T} e^{j\omega t}$$

The identity of (16) and (17) shows that (15) is the correct solution of the problem under the assumptions involved.

IMPEDANCE OF A TOROID CONTAINING A LAMINATED CORE

7. If the cross section of the winding of a toroid is small compared to the large radius, the field distribution can be considered to be the same as that in a long solenoid and expression (15) may be employed for a toroid containing laminations of thickness small compared to their width. Using (15) the flux ϕ' contained in one lamination of a toroid is

$$\phi' = \int B\,da = \mu W \int_{-T}^{+T} H\,dy = \frac{2\mu W n I}{\alpha} e^{j\omega t} \operatorname{Tanh} \alpha T \qquad (18)$$

The flux ϕ'' in all the laminations (for laminations of equal dimensions and properties) is

$$\phi'' = \frac{2m\mu W n I}{\alpha} e^{j\omega t} \operatorname{Tanh} \alpha T \qquad (19)$$

where m is the total number of laminations.

The combined cross section area of the lamination plus the leakage area A_0 equals the cross section area A' of the winding, i.e.

$$A' = 2mTW + A_0 = KA' + (1-K)A' \qquad (20)$$

where K is a constant defined as

$$K = \frac{\text{Total Lamination Area}}{\text{Total Cross Section Area of Winding}} = \frac{2mTW}{A'} \qquad (21)$$

From (15) the total flux ϕ in the toroid is then

$$\phi = \phi'' + (1-K)A'\mu_0 n I e^{j\omega t}$$

$$= \frac{\mu A' n I e^{j\omega t}}{\alpha T} K \operatorname{Tanh} \alpha T + (1-K) A' \mu_0 n I e^{j\omega t} \qquad (22)$$

If the D.C. toroid resistance is negligible, the voltage drop v across the toroid is

$$v = N\frac{d\phi}{dt} = \left[\frac{\mu A' N n K}{\alpha T}\tanh \alpha T + (1-K)\mu_0 A' N n\right] j\omega I e^{j\omega t} \tag{23}$$

The air core inductance L_0 of the toroid is

$$L_0 = \mu_0 N A' n \tag{24}$$

with which (23) can be written as

$$v = \left[\frac{k_m K \tanh \alpha T}{\alpha T} + 1 - K\right] j\omega L_0 I e^{j\omega t} \tag{25}$$

For high permeability laminations or for K = 1

$$\frac{k_m K \tanh \alpha T}{\alpha T} \gg 1 - K$$

and the expression for the voltage across the toroid becomes

$$v = j\omega L_0 \left(\frac{k_m K \tanh \alpha T}{\alpha T}\right) I e^{j\omega t} \tag{26}$$

Because α is complex we will express (26) in a more significant form as

$$v = \frac{\sqrt{2}\, k_m K \omega L_0}{\delta T}\left|\frac{\cosh \delta T - \cos \delta T}{\cosh \delta T + \cos \delta T}\right|^{\frac{1}{2}} I e^{j\omega t} \Big/ \tan^{-1}\frac{\sinh \delta T + \sin \delta T}{\sinh \delta T - \sin \delta T} \tag{27}$$

where $\delta = \sqrt{2\sigma\mu\omega}$

v in e.m.u. is

$$v = \frac{\sqrt{2}\, K\mu\omega L_0}{\delta' T}\left|\frac{\cosh \delta' T - \cos \delta' T}{\cosh \delta' T + \cos \delta' T}\right|^{\frac{1}{2}} I e^{j\omega t} \Big/ \tan^{-1}\frac{\sinh \delta' T + \sin \delta' T}{\sinh \delta' T - \sin \delta' T} \tag{28}$$

where $\delta' = \sqrt{8\pi\sigma\mu\omega}$

The impedance Z of the toroid in M.K.S. units and e.m.u. is, respectively,

$$Z_{(MKS)} = \frac{V}{I} = \frac{\sqrt{2}\, k_m K \omega L_o}{\delta T} \left[\frac{\cosh \delta T - \cos \delta T}{\cosh \delta T + \cos \delta T} \right]^{\frac{1}{2}} \Big/ \tan^{-1} \frac{\sinh \delta T + \sin \delta T}{\sinh \delta T - \sin \delta T} \quad (29)$$

$$Z_{(emu)} = \frac{V}{I} = \frac{\sqrt{2}\, K \mu \omega L_o}{\delta' T} \left[\frac{\cosh \delta' T - \cos \delta' T}{\cosh \delta' T + \cos \delta' T} \right]^{\frac{1}{2}} \Big/ \tan^{-1} \frac{\sinh \delta' T + \sin \delta' T}{\sinh \delta' T - \sin \delta' T} \quad (30)$$

The resistance and inductive reactance of the toroid are, of course, the real and imaginary parts, respectively, of the above expressions and both are functions of the frequency, permeability, conductivity and lamination thickness. If we call the real part of (29) or (30) R_e, the effective resistance, and the imaginary part ωL_e, the effective inductive reactance, the effective impedance is

$$R_e + j\omega L_e \quad (31)$$

This effective impedance is equivalent to that obtained by K. L. Scott (Ref. 4). The total impedance Z_T of the toroid is equivalent to (31) in series with the d.c. resistance R_{dc} of the toroid winding or

$$Z_T = R_{DC} + R_e + j\omega L_e \quad (32)$$

8. It was desired to experimentally check (29) and (30) with a laminated core of constant permeability and no hysteresis effect which means that such a core would have to be non-ferromagnetic. An experimental check on the above expressions should be performed on laminations which have a thickness small compared to the lamination width. The toroids that could be made at the Naval Ordnance Laboratory, which would not involve an excessive amount of expense, material and time, would require a lamination of a maximum thickness of about 25 mils in order that the assumptions made in the above expressions approximately represent the actual physical case. However, if a copper, aluminum, or

other readily available non-ferromagnetic materials of constant permeability (i.e. $\mu = 1$) were used for laminations, the low permeability characteristic of these materials would cause relatively small eddy current losses as compared to those induced in high permeability ferromagnetic materials even though the conductivity of the former might be several times greater than the latter. Because of the small eddy current effect encountered by using non-ferromagnetic laminations of reasonable thicknesses or should we say, "thinnesses", changes in the resistance and inductance would be difficult to detect at frequencies low enough so that capacity effects would not be introduced to complicate matters. Consequently, a toroid with a solid circular core was constructed to experimentally check the following theory.

IMPEDANCE OF A TOROID HAVING A SOLID ROUND COPPER CORE

9. If R is the cross-section radius of the solid toroid core and r is the distance from the core center, the field distribution in the core will become equivalent to that in a core of infinite length and of the same radius R and core material if the radius of the toroid is large compared to the cross-section radius. If the axis of the long cylindrical core lies along the Z axis, the field will be independent of Z and the Laplacian of (8) becomes equal to

$$\nabla^2 H = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial H}{\partial r}\right) = \frac{\partial^2 H}{\partial r^2} + \frac{1}{r}\frac{\partial H}{\partial r} = \sigma\mu\frac{\partial H}{\partial t} \quad \text{(MKS)} \qquad (33)$$

The separation of variables method gives Bessel's differential equation

$$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + C^2 R = 0 \qquad (34)$$

where R is the expression for the field as a function of r and $C^2 = -j\sigma\mu\omega$. (34) has the solution

$$H = H_o e^{j\omega t} J_o(cr) \qquad (35)$$

and (35) represents an infinite series of the form

$$H = H_0 e^{j\omega t}\left(1 - \frac{c^2 r^2}{2^2} + \frac{c^4 r^4}{2^4(2!)^2} - \frac{c^6 r^6}{2^6(3!)^2} + \frac{c^8 r^8}{2^8(4!)^2} \cdots\right)$$

$$= H_0 e^{j\omega t}\left[\left(1 - \frac{(\epsilon r)^4}{2^4(2!)^2} + \frac{(\epsilon r)^8}{2^8(4!)^2} \cdots\cdots\right) + j\left(\frac{(\epsilon r)^2}{2^2} - \frac{(\epsilon r)^6}{2^6(3!)^2} + \frac{(\epsilon r)^{10}}{2^{10}(5!)^2} \cdots\cdots\right)\right] \qquad (36)$$

where $\epsilon^2 = \sigma\mu\omega$. The field H_R at the boundary of r = R is

$$H_R = nI_0 e^{j\omega t} = H_0 e^{j\omega t}\left[\alpha + j\beta\right] \qquad (37)$$

and consequently α and β are the ber and bei functions of ϵR, respectively. Hence

$$H = \frac{nI_0}{\alpha + j\beta} e^{j\omega t}\left[\left(1 - \frac{(\epsilon r)^4}{2^4(2!)^2} + \frac{(\epsilon r)^8}{2^8(4!)^2} \cdots\cdots\right) + j\left(\frac{(\epsilon r)^2}{2^2} - \frac{(\epsilon r)^6}{2^6(3!)^2} + \frac{(\epsilon r)^{10}}{2^{10}(5!)^2} \cdots\right)\right] \qquad (38)$$

The flux ϕ therefore is

$$\phi = \mu \int_0^R 2\pi r H \, dr = 2\pi\mu R^2 n I_0 e^{j\omega t} \frac{\delta + j\gamma}{\alpha + j\beta} \qquad (39)$$

where

$$\alpha = 1 - \frac{(\epsilon R)^4}{2^4(2!)^2} + \frac{(\epsilon R)^8}{2^8(4!)^2} - \frac{(\epsilon R)^{12}}{2^{12}(6!)^2} + \frac{(\epsilon R)^{16}}{2^{16}(8!)^2} \cdots\cdots$$

$$\beta = \frac{(\epsilon R)^2}{2^2} - \frac{(\epsilon R)^6}{2^6(3!)^2} + \frac{(\epsilon R)^{10}}{2^{10}(5!)^2} - \frac{(\epsilon R)^{14}}{2^{14}(7!)^2} + \cdots$$

$$\delta = \frac{1}{2} - \frac{(\epsilon R)^4}{6 \cdot 2^4(2!)^2} + \frac{(\epsilon R)^8}{10 \cdot 2^8(4!)^2} - \frac{(\epsilon R)^{12}}{14 \cdot 2^{12}(6!)^2} + \frac{(\epsilon R)^{16}}{18 \cdot 2^{16}(8!)^2}$$

$$\gamma = \frac{(\epsilon R)^2}{4 \cdot 2^2} - \frac{(\epsilon R)^6}{8 \cdot 2^6(3!)^2} + \frac{(\epsilon R)^{10}}{12 \cdot 2^{10}(5!)^2} - \frac{(\epsilon R)^{14}}{16 \cdot 2^{14}(7!)^2} + \cdots \qquad (40)$$

The voltage v across the toroid is

$$v = N\frac{d\Phi}{dt} = 2j\omega L_0 I e^{j\omega t}\,\frac{\delta + j\gamma}{\alpha + j\beta}$$

$$= \frac{2j\omega L_0 I}{\alpha^2+\beta^2}\, e^{j\omega t}\left[\alpha\delta + \beta\gamma + j(\alpha\gamma - \beta\delta)\right] \qquad (41)$$

and the effective resistance R_e' and inductance L_e' of the toroid with no leakage space between the core and the winding are

$$R_e' = 2\omega L_0\left(\frac{\beta\delta - \alpha\gamma}{\alpha^2+\beta^2}\right)$$

$$L_e' = 2L_0\left(\frac{\alpha\delta + \beta\gamma}{\alpha^2+\beta^2}\right) \qquad (42)$$

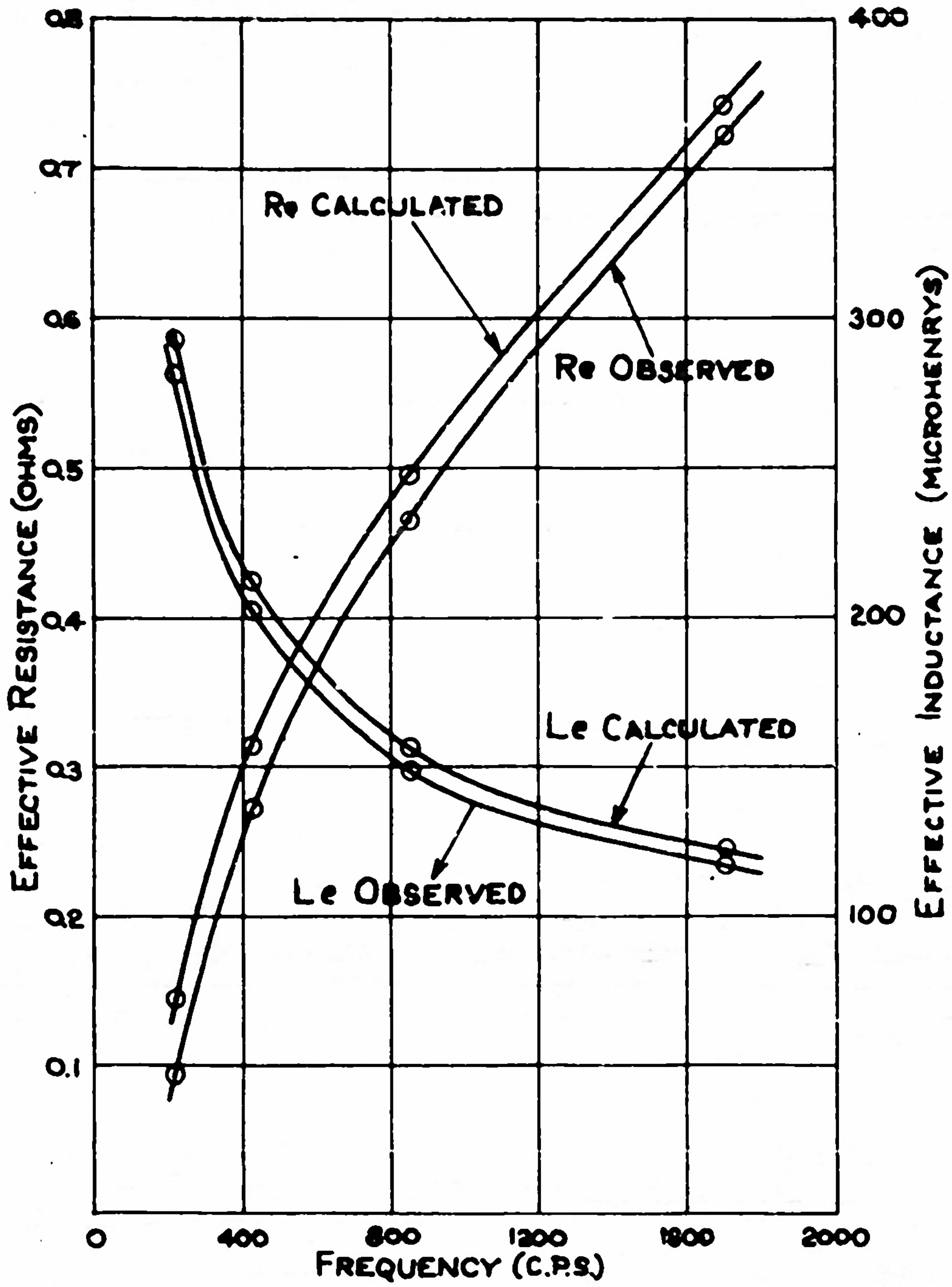
But if there is leakage space and the ratio of the cross-section of the solid core to the cross section of the winding is called K, then the effective resistance R_e and effective inductance L_e are

$$R_e = 2\omega K L_0\left(\frac{\beta\delta - \alpha\gamma}{\alpha^2+\beta^2}\right) \qquad \text{(M.K.S.)} \qquad (43)$$

$$L_e = 2KL_0\,\frac{\alpha\delta + \beta\gamma}{\alpha^2+\beta^2} + (1-K)L_0 \qquad \text{(M.K.S.)} \qquad (44)$$

Figure 3 shows the theoretical and experimental values of the effective resistance and inductance of a toroid about 12 inches in diameter with a 1/2" diameter solid annealed copper core having an assumed conductivity of .591 X 10^6 mhos/cm.

9. The observed and calculated values of the effective inductance come within 5% of each other while the observed



PLOT OF THE EFFECTIVE RESISTANCE AND INDUCTANCE VS. FREQUENCY OF A 12" DIAMETER TOROID HAVING A SOLID ANNEALED COPPER CORE OF A 1/2" DIAMETER

Fig. 3

and calculated resistance agree to within 13% except at 2?2 C.P.S. point, at which frequency the bridge employed does not give the most accurate experimental measurements. These results are in fair agreement if we remember that the conductivity of copper was assumed to be that given in the Physics Handbook and that all of the flux was assumed to lie within the toroid winding. The purpose of the work connected with figure 3 was to show that the absolute values of inductance and resistance can be accurately calculated if the core permeability is constant and if there is no hysteresis effect present. We shall now continue with (27) and (29) to find the expression for the Epstein test power loss per unit volume of core material of thin laminations having a permeability which is assumed to be constant throughout the core.

POWER LOSS IN NON-FERROMAGNETIC LAMINATED CORES

11. Neglecting the d.c. winding resistance, the power loss P of the toroid is

$$P = \frac{V_m I_m}{2} \cos\theta \tag{45}$$

where V_m and I_m are the amplitudes of the voltage and current, respectively, and θ is the phase angle between the voltage and current. From (28), V_m in e.m.u. is

$$V_m = \frac{\sqrt{2}\, K \mu \omega L}{\delta' T} \left[\frac{\cosh \delta' T - \cos \delta' T}{\cosh \delta' T + \cos \delta' T}\right] I_m \tag{46}$$

By substitution of I_m obtained from (46), (45) becomes:

$$P = \frac{V_m^2\, \delta' T}{2\sqrt{2}\, K \mu \omega L} \left[\frac{\cosh \delta' T + \cos \delta' T}{\cosh \delta' T - \cos \delta' T}\right]^{\frac{1}{2}} \cos\theta \tag{47}$$

where $$\cos\theta = \frac{\sinh \delta' T - \sin \delta' T}{\sqrt{2}\sqrt{\sinh^2 \delta' T + \sin^2 \delta' T}} = \frac{\sinh \delta' T - \sin \delta' T}{\sqrt{2}\sqrt{\cosh^2 \delta' T - \cos^2 \delta' T}} \tag{48}$$

and by simplification

$$P = \frac{V_m \delta' T}{4 K \mu \omega l_m} \left[\frac{\sinh \delta' T - \sin \delta' T}{\cosh \delta' T - \cos \delta' T} \right] \tag{49}$$

For toroids having K = 1 or for cores in which the permeability is very much greater than 1, the "effective" induction $\bar{B}_m$ is defined as the maximum amplitude of the sinusoidally varying flux divided by the core cross-sectional area and is given by the relations

$$V_m = N \omega K \bar{B}_m A' = N \omega \phi_m$$

$$\bar{B}_m = \phi / K A' \tag{50}$$

where A and K were previously defined by (20) and (21). The expression for the power loss in terms of $\bar{B}_m$ is then

$$P = \frac{\omega N^2 K \bar{B}_m^{\,2} A'^2 \delta' T}{4 \mu l_m} \left[\frac{\sinh \delta' T - \sin \delta' T}{\cosh \delta' T - \cos \delta' T} \right]$$

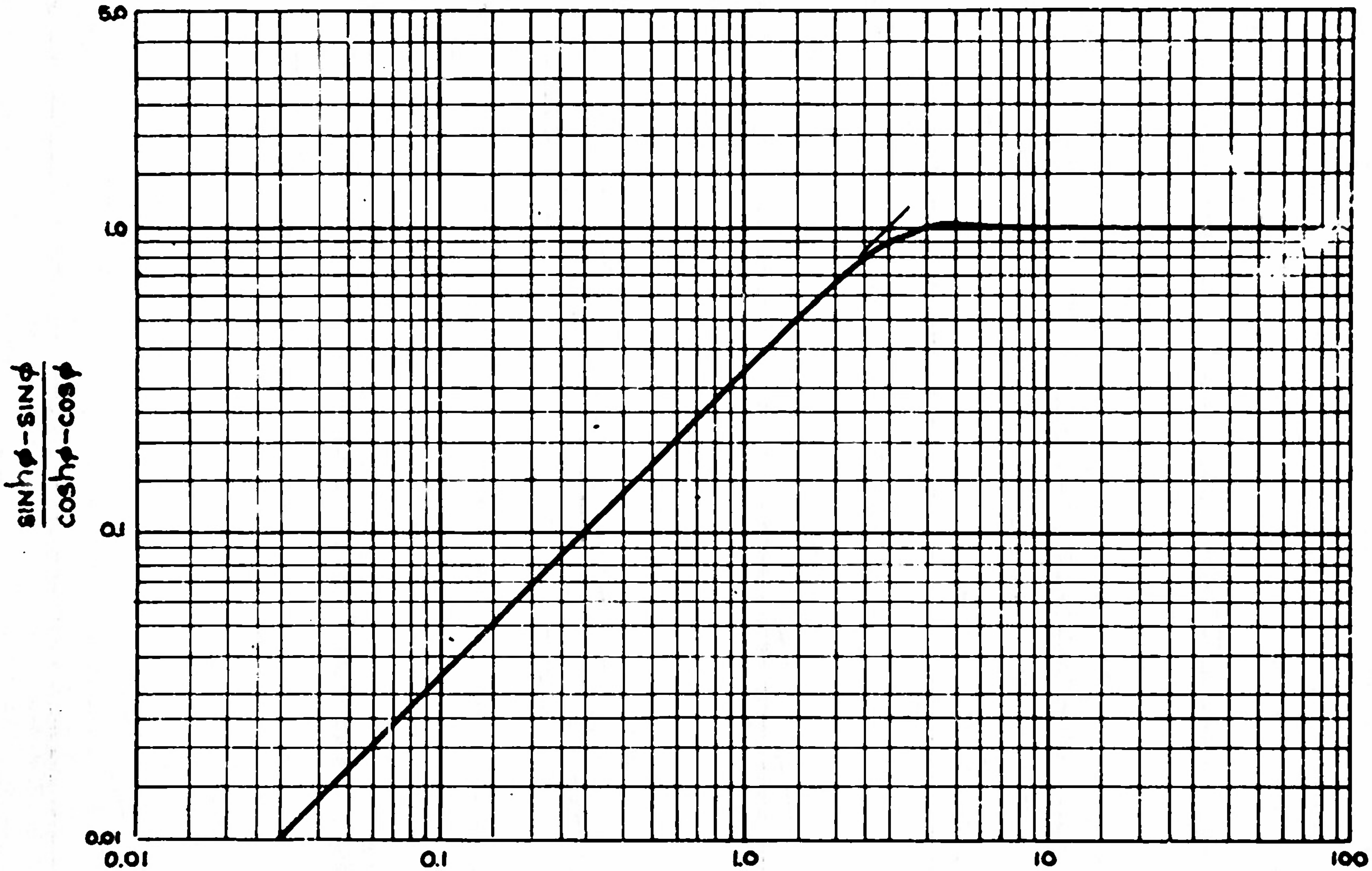
$$= \frac{\omega N^2 K \bar{B}_m^2 \delta' T A' l}{16 \pi \mu N^2} \left[\frac{\sinh \delta' T - \sin \delta' T}{\cosh \delta' T - \cos \delta' T} \right] \tag{51}$$

where l is the mean magnetic length of the toroid. Dividing both sides of (51) by Al, the expression we obtain for the power loss per unit volume of a core material is

$$\frac{P}{V} = \frac{\omega K \bar{B}_m^2 \sqrt{8 \pi \sigma \mu \omega}\, T}{16 \pi \mu} \left[\frac{\sinh \delta' T - \sin \delta' T}{\cosh \delta' T - \cos \delta' T} \right] \tag{52}$$

Figure 4 shows $(\sinh \delta' T - \sin \delta' T) / (\cosh \delta' T - \cos \delta' T)$

plotted as a function of $\delta' T = \sqrt{8 \pi \sigma \mu \omega}\, T = \phi$ which may be approximated by drawing two straight lines which intersect at $\delta' T = 3$. Consequently, for $\delta' T < 3$,



$\phi = \delta' T = \sqrt{8\pi\sigma\mu\omega}\ T$

Fig. 4

$$\frac{\sinh\delta' T - \sin\delta' T}{\cos\delta' T - \cos\delta' T} = \frac{\delta' T}{3} \tag{53}$$

and for $\delta' T > 3$

$$\frac{\sinh\delta' T - \sin\delta' T}{\cos\delta' T - \cos\delta' T} = 1 \tag{54}$$

This may be summarized in e.m.u. as follows:

For $\sqrt{8\pi\sigma\mu\omega}\, T = \phi < 3$

$$\frac{P}{V} = \frac{K\overline{B}_m^2\omega^2 T^2\sigma}{6} = \frac{K\pi^2\overline{B}_m^2 f^2 d^2\sigma}{6} \tag{55}$$

where the lamination thickness $d = 2T$

For $\sqrt{8\pi\sigma\mu\omega}\, T = \phi > 3$

$$\frac{P}{V} = \frac{\pi K\overline{B}_m^2 f^{3/2}\sigma^{1/2} d}{4\mu^{1/2}} \tag{56}$$

12. Bozorth (Ref. 1) gives for the power loss

$$\frac{P}{V} = \frac{\pi^2 d^2 B^2 f^2\sigma}{6} \quad \text{for } \phi < 1 \tag{57}$$

$$\frac{P}{V} = \frac{\pi B^2 f^{3/2} d\sigma^{1/2}}{2\mu^{1/2}} = \frac{B^2 f^{1/2}}{8\pi\sigma\mu^{3/2} d} \quad \text{for } \phi >> 1 \tag{58}$$

It should be pointed out that B of (57) is not the same as B' of (58). In (57), B is the induction when flux penetration is complete and hence is the same as our $\overline{B}_m$ in (55) so that these two expressions agree. In (58), B' is the maximum effective induction at the edge of the sheet and is defined as $\mu' H_b$ where μ' is the effective permeability and H_b is the amplitude of the sinusoidal field at the lamination boundary. However, our $\overline{B}_m$ is defined as the effective induction obtained by averaging the maximum amplitude of the induction over the area of the sheet and hence can be expected to be different from B'. In fact, Bozorth shows that if μ is the actual value of the constant permeability, then when ϕ is large, $\overline{B}_m = \frac{\sqrt{2}}{\phi}\mu H_b$

and $\mu' = \frac{\mu}{\phi}$. This leads at once to $\overline{B}_m = \sqrt{2}\,\mu' H_b = \sqrt{2}\,B'$

so that (56) immediately reduces to (58) as it should. Also, inspection of figure 4 shows that the critical value of ϕ can be more definitely specified than those of (57) and (58).

CONCLUSION

13. We have seen how (55) and (56) are derived by assuming the laminated core material to have a constant permeability. Further eddy current analysis was attempted for a hypothetical core material having a variable permeability with no hysteresis effect. A simple relation was assumed for a variable permeability i.e., $\mu \propto |H|$ (this is roughly true for some materials at the lower portion of the magnetization curve). However, no solution was obtained because of the complexity of the partial differential equation and the boundary conditions to be satisfied. It is doubtful that its solution for the power loss, if possible, would be equal to (55) and (56). By this reasoning it is concluded that (55) and (56) are intended for non-ferromagnetic materials and if they are applied to calculate the power loss in laminated ferromagnetic cores, an anomaly will result.

ACKNOWLEDGEMENT

14. The author wishes to thank Mr. D. I. Gordon for his guidance and assistance rendered in the writing of this report. Many thanks are also owed to Dr. R. K. Wangsness and Dr. J. McFadden for reviewing the text and for their constructive criticisms and helpful suggestions.

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